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ON THE STRESSES IN A NONLINEAR BEAM SUBJECT TO RANDOM EXCITATION

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Abstract—The technique of equivalent linearization is used to investigate the effect of the membrane force on the stresses in a simply supported Bernoulli–Euler beam undergoing moderately large random vibrations. It is shown that the percentage reduction of the mean square bending stress can be substantially less than the percentage reduction of the mean square displacements, thereby lowering any expected 'nonlinear safety factor' of the stresses. Furthermore, the difference of percentage reduction is greater for wider spectral densities of the load.

INTRODUCTION

THE response of linear continuous structures to random excitation has received much attention in the past few years [1–5]. Eringen [2] has shown that the series for the mean square stresses in a Bernoulli–Euler beam subject to a purely random excitation diverge. Bogdanoff and Goldberg [6] have since shown that a more realistic type of loading produces finite mean square stresses.

Since the linear theory of structures is generally valid for only very small displacements, the response of nonlinear structures to random excitation has recently been investigated [7–10]. Lin [8] has used the method of equivalent linearization to investigate the single mode response of a slightly nonlinear panel. He has indicated that the mean square response of the panel is reduced due to the membrane forces. Herbert [9, 10] has since used the method of the Markoff process and the associated Fokker–Planck equation to investigate the response of nonlinear beams and plates to purely random loadings. He has shown that the nonlinear coupling of the modes can be quite significant.

In considering the response of nonlinear structures to random excitation it is important to ascertain the effect of the nonlinearity on the stresses. Since the displacements of a beam are considerably reduced due to the membrane force we should expect the same result for the stresses. However, the question arises as to whether this reduction is the same as that of the displacements. It was impossible in reference [9] to investigate this problem since, for purely random loadings, the mean square stresses diverge. Therefore, it was deemed advisable to study the response of a nonlinear Bernoulli–Euler beam subject to a realistic random loading, i.e. one with finite power. This paper is the result of such an investigation. Because the method of the Markoff process and the associated Fokker–Planck equation is only applicable to systems excited by white noise, the technique of equivalent linearization is used. While this technique is valid for only slightly nonlinear systems, it still gives us some insight as to the effect of the nonlinearity on the stresses.

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ANALYSIS

Consider an elastic beam with pin-ended supports which are restrained from motion. Then the equation governing moderately large vibrations is

$$EI\frac{\partial^4 w}{\partial x^4}+N\frac{\partial^2 w}{\partial x^2}+\rho A\frac{\partial^2 w}{\partial t^2}+\beta\frac{\partial w}{\partial t} = q(x, t), \tag{1}$$

where the membrane force N is given by

$$N = \frac{EA}{2L}\int_0^L \left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}x \tag{2}$$

and E is the elastic modulus, I the moment of inertia of the cross section, ρ the mass density, A the cross-sectional area, L the length of the beam and q the random load per unit length acting on the beam. We expand w and q in terms of the eigenfunctions of the linear problem so that we have

$$w(x, t) = \sum_{m=1}^{\infty} w_m(t)\sin\frac{m\pi x}{L}, \tag{3}$$

$$q(x, t) = \sum_{m=1}^{\infty} q_m(t)\sin\frac{m\pi x}{L}. \tag{4}$$

Upon substitution of (3) and (4) into (1) and (2) we obtain the following equation governing the w_m:

$$\ddot{w}_m+\beta_0\dot{w}_m+\omega_m^2\left(1+\frac{1}{4R^2m^2}\sum_{n=1}^{\infty} n^2w_n^2\right)w_m = a_m \tag{5}$$

where

$$\beta_0 = \frac{\beta}{\rho A}, \quad a_m = \frac{q_m}{\rho A}, \quad \omega_m^2 = \frac{\pi^4 EIm^4}{\rho AL^4}, \quad R^2 = \frac{I}{A},$$

and

$$q_m(t) = \frac{2}{L}\int_0^L q(x, t)\sin\frac{m\pi x}{L}\,\mathrm{d}x. \tag{6}$$

Equation (5) is a nonlinear stochastic differential equation. Knowing statistical properties of the driving function a_m, there exists no standard technique for obtaining the statistical properties of the response variable w_m. For small nonlinearities we can obtain approximate values of these statistical properties by using the technique of equivalent linearization.

We rewrite (5) as

$$\ddot{w}_m+\beta_0\dot{w}_m+k_m\omega_m^2 w_m+\varepsilon_m = a_m \tag{7}$$

where

$$\varepsilon_m = \omega_m^2 w_m \left(1 - k_m + \frac{1}{4R^2m^2} \sum_{n=1}^{\infty} n^2 w_n^2\right). \tag{8}$$

If in (7) ε_m is neglected then (7) will be linear and can be handled by established techniques. Obviously the error will be smaller if ε_m is smaller. Therefore the choice of k_m is the value which minimizes some statistical measure of ε_m. The most mathematically expedient measure is the mean square value. This is given by

$$\overline{\varepsilon_m^2} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \omega_m^4 w_m^2 \left(1 - k_m + \frac{1}{4R^2m^2} \sum_{n=1}^{\infty} n^2 w_n^2\right)^2 \mathrm{d}t. \tag{9}$$

For an ergodic process the time average may be replaced by an ensemble average so that (9) may be replaced by

$$\overline{\varepsilon_m^2} = \omega_m^4 \left\langle w_m^2 \left(1 - k_m + \frac{1}{4R^2m^2} \sum_{n=1}^{\infty} n^2 w_n^2\right)^2 \right\rangle. \tag{10}$$

If the load is assumed to be Gaussian then with ε_m neglected in (7) the w_m will be Gaussian and distributed according to the law

$$p(w_m) = \frac{1}{(2\pi)^{\frac{1}{2}} \sigma_m} \exp\left(-\frac{w_m^2}{2\sigma_m^2}\right) \tag{11}$$

where

$$\sigma_m^2 = \langle w_m^2 \rangle. \tag{12}$$

Equation (10) may then be written as

$$\overline{\varepsilon_m^2} = \omega_m^4 \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} w_m^2 \left(1 - k_m + \frac{1}{4R^2m^2} \sum_{n=1}^{\infty} n^2 w_n^2\right)^2 \prod_{r=1}^{\infty} \frac{1}{(2\pi)^{\frac{1}{2}} \sigma_r} \exp\left(-\frac{w_r^2}{2\sigma_r^2}\right) \mathrm{d}w_r. \tag{13}$$

Minimizing $\overline{\varepsilon_m^2}$ with respect to k_m leads to

$$k_m = \frac{\displaystyle\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(1 + \frac{1}{4R^2m^2} \sum_{n=1}^{\infty} n^2 w_n^2\right) w_m^2 \prod_{r=1}^{\infty} \exp\left(-\frac{w_r^2}{2\sigma_r^2}\right) \mathrm{d}w_r}{\displaystyle\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} w_m^2 \prod_{r=1}^{\infty} \exp\left(\frac{w_r^2}{2\sigma_r^2}\right) \mathrm{d}w_r}. \tag{14}$$

Evaluating the integrals,

$$k_m = 1 + \frac{1}{4R^2m^2} \left(\sum_{n=1}^{\infty} n^2 \sigma_n^2 + 2m^2 \sigma_m^2\right). \tag{15}$$

If in (7) ε_m is neglected and we assume that

$$\langle q(x, t) q(y, t + \tau) \rangle = \delta(x - y) R(\tau) \tag{16}$$

then

$$\sigma_m^2 = \frac{2}{\rho^2 A^2 L}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_m(\tau_1)h_m(\tau_2)R(\tau_1-\tau_2)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2 \tag{17}$$

where

$$h_m(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{j\omega\tau}\,\mathrm{d}\omega}{\omega_m^2 k_m-\omega^2+j\beta_0\omega}. \tag{18}$$

We wish $R(\tau)$ to be such that the total power of the input is finite. That is, we want

$$\int_{-\infty}^{\infty} A(\omega)\,\mathrm{d}\omega < \infty, \tag{19}$$

where

$$A(\omega) = \int_{-\infty}^{\infty} R(\tau)e^{-j\omega\tau}\,\mathrm{d}\tau. \tag{20}$$

A simple form which has this feature is

$$R(\tau) = N_0 L\omega_0 e^{-|\omega_0\tau|}, \tag{21}$$

$$A(\omega) = \frac{N_0 L\omega_0^2}{\omega_0^2+\omega^2}. \tag{22}$$

While this spectral density is not necessarily a typical form for real loadings it has the features of finite power, one adjustable parameter and ease in performing the ensuing integrations.

Combining (17), (18) and (21) leads us to the following results for σ_m^2:

$$\sigma_m^2 = \frac{N_0}{\rho^2A^2}\left[\frac{\omega_0+\dfrac{\omega_0^2}{\beta_0}\left(1+\dfrac{\omega_0^2}{k_m\omega_m^2}-\dfrac{\beta_0^2}{k_m\omega_m^2}\right)}{(k_m\omega_m^2+\omega_0^2)^2-\beta_0^2\omega_0^2}\right]. \tag{23}$$

If we consider the important case of light damping for which

$$\beta_0 \ll \omega_m,$$
$$\beta_0 \ll \omega_0,$$

then (23) reduces to

$$\sigma_m^2 = \frac{\mu\sigma_0^2}{k_m m^4(m^4k_m+\mu)}, \tag{24}$$

where

$$\mu = \frac{\omega_0^2}{\omega_1^2}, \qquad \sigma_0^2 = \frac{N_0L^4}{\beta EI\pi^4}. \tag{25}$$

It should be noted that the limiting case of white noise can be obtained by letting $\omega_0 \to \infty$. That is,

$$\lim_{\omega_0 \to \infty} R(\tau) = N_0 L \delta(\tau), \tag{26}$$

$$\lim_{\omega_0 \to \infty} A(\omega) = N_0 L. \tag{27}$$

For this case equation (24) reduces to

$$\sigma_m^2 = \frac{\sigma_0^2}{k_m m^4}, \tag{28}$$

and for $k_m = 1$, the linear case, this reduces to a well-known result.

Equations (15) and (24) are the two equations governing k_m and σ_m^2. Elimination of k_m between these two equations would lead us to σ_m^2. Unfortunately, this is not such a simple process. An alternate approach is to compute the k_m on the basis that σ_m^2 is that of the linear problem. The accuracy of this approach is of the same order of magnitude as that obtained by solving equations (15) and (24) exactly, [11].

Thus, since the linear value of σ_m^2 is given by

$$(\sigma_m^2)_{\text{linear}} = \frac{\mu \sigma_0^2}{m^4(m^4+\mu)}, \tag{29}$$

then k_m is given by

$$k_m = 1 + \frac{\sigma_0^2}{4R^2}\left[\frac{F(\mu)}{m^2} + \frac{2\mu}{m^4(m^4+\mu)}\right], \tag{30}$$

where

$$F(\mu) = \sum_{m=1}^{\infty} \frac{\mu}{m^2(m^4+\mu)}. \tag{31}$$

Substitution of (30) and (31) into (24) would lead us to σ_m^2.

Having σ_m^2, the various mean square quantities characterizing the response of the beam can be computed. For example, the mean square displacement of the beam is simply

$$\langle w^2(x,t) \rangle = \sum_{m=1}^{\infty} \sigma_m^2 \sin^2 \frac{m\pi x}{L}, \tag{32}$$

which can be seen to be less than the corresponding displacement of the linear problem.

The bending stress in the beam is given by

$$S_B = \frac{Eh}{2}\frac{\partial^2 w}{\partial x^2} = \frac{Eh}{2}\sum_{m=1}^{\infty} m^2 w_m \sin \frac{m\pi x}{L} \tag{33}$$

and the membrane stress by

$$S_M = \frac{N}{A} = \frac{E}{2L}\int_0^L \left(\frac{\partial w}{\partial x}\right)^2 dx = \frac{E\pi^2}{4L^2}\sum_{m=1}^{\infty} m^2 w_m^2. \tag{34}$$

The total mean square stress is

$$\langle S^2\rangle = \langle (S_B+S_M)^2\rangle = \langle S_B^2\rangle + 2\langle S_M S_B\rangle + \langle S_M^2\rangle. \tag{35}$$

Now from (11), (33) and (34)

$$\langle S_M S_B\rangle = 0, \tag{36}$$

so that

$$\langle S^2\rangle = \langle S_M^2\rangle + \langle S_B^2\rangle \tag{37}$$

or

$$\langle S^2\rangle = \frac{E^2h^2\pi^4}{4L^4}\left(\sum_{m=1}^{\infty} m^4\sigma_m^2 \sin^2\frac{m\pi x}{L} + \frac{1}{4h^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} m^2n^2\langle w_m^2 w_n^2\rangle\right). \tag{38}$$

which upon making use of (11) and (24) reduces to

$$\langle S^2\rangle = \frac{E^2h^2\pi^4}{4L^4}\sigma_0^2\mu\Bigg[\sum_{m=1}^{\infty} k_m^{-1}(m^4k_m+\mu)^{-1}\sin^2\frac{m\pi x}{L} + \frac{\sigma_0^2\mu}{4h^2}\left(\left\{\sum_{m=1}^{\infty}[k_m m^2(m^4+\mu)]^{-1}\right\}^2 + 2\sum_{m=1}^{\infty}[k_m m^2(m^4+\mu)]^{-2}\right)\Bigg], \tag{39}$$

where k_m is given by equation (24).

NUMERICAL RESULTS

As previously mentioned it is to be expected that the membrane force should cause a reduction of the mean square displacements and stresses. Indeed, inspection of equations (32) and (39) verifies this. However, we wish to know if the reduction is the same for both.

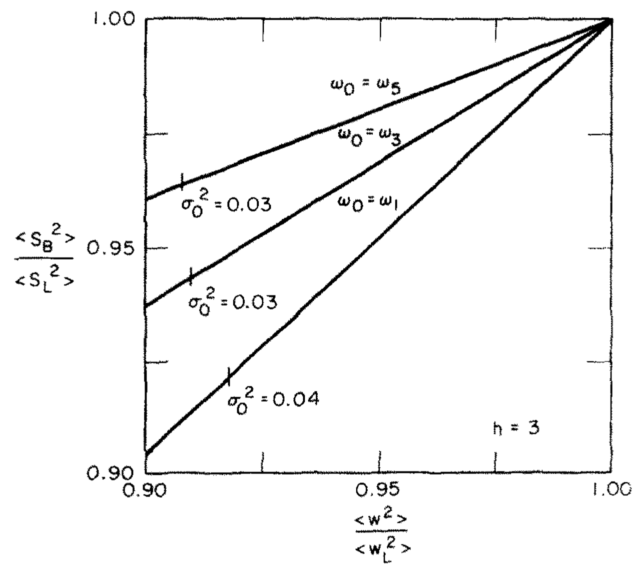

FIG. 1. Ratio of reduction of stress to reduction of deflection at midspan.

To answer this question numerical computations have been performed for the case of a rectangular beam with $h = 3$, σ_0^2 ranging from 0 to 0·04, and for various values of ω_0 (h and σ_0 carry the dimensions of length).

In performing the calculations it was found that while the membrane force reduces $\langle S_B^2 \rangle$, the difference between $\langle S^2 \rangle$ and $\langle S_B^2 \rangle$ is negligible, for the range of parameters considered. The significant results of the computations are presented in Fig. 1. There the ratio $\langle S_B^2 \rangle / \langle S_L^2 \rangle$ is plotted against the ratio $\langle w^2 \rangle / \langle w_L^2 \rangle$ where $\langle S_L^2 \rangle$ and $\langle w_L^2 \rangle$ are the mean square stress and displacement, respectively, of the linear beam. It can be seen from these curves that for $\mu = 1$ ($\omega_0 = \omega_1$) the percentage reduction of mean square stress and mean square displacement are nearly equal. However, upon increasing ω_0 a smaller percentage reduction of the mean square stress occurs. As the driving frequency continues to increase the difference in percentage reduction of the stress and displacement becomes greater.

Designers and analysts tend to think they have a 'nonlinear safety factor' in problems such as these. While this is certainly true, these computations indicate that higher driving frequencies reduce this effect on the stresses so that structures may not be as safe as anticipated.

CONCLUSIONS

This paper has been devoted to the investigation of the effects of the membrane force on the stresses in a simply supported Bernoulli–Euler beam undergoing moderately large random vibrations. The method of equivalent linearization has been used to derive approximate expressions for mean square displacements and stresses.

Numerical computations have indicated that the percentage reduction of the mean square stresses can be substantially less than the percentage reduction of the mean square displacements thereby reducing any 'nonlinear safety factor' that one might consider. Furthermore, as the spectral density of the load becomes wider the difference in the percentage reduction of stress and displacement becomes greater.

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Zusammenfassung—Das Verfahren der äquivalenten Linearisierung wird dazu herangezogen, die Wirkung der Membrankraft auf die Spannungen in einem einfachen, gestützen Benoulli–Euler Balken zu untersuchen, der mässig starken Zufallsschwingungen ausgestzt ist. Es wird gezeigt, dass die prozentuale Senkung des mittleren Quadratwertes der Biegebeanspruchung wesentlich unter der prozentualen Senkung des mittleren Quadratwertes der Auslenkungen liegen kann, wodurch der erwartete "nichtlineare Sicherheitsfaktor" der Beanspruchungen herabgesetzt wird. Ausserdem vergrössert sich bei breiteren Sprektraldichten der Belastung der Unterschied zwischen den prozentualen Senkungen.

Абстракт—Воспользуется метод эквивалентной линеаризации для исследования воздействия мембранной силы на напряжения в свободно опёртой балке Бернулли-Эйлера при умеренно больших случайных колебаниях. Показывается, что относительное уменьшение среднеквадратичного изгибающего напряжения может быть существенно меньше относительного уменьшения среднеквадратичных перемещений, что приводит к снижению какого-либо ожидаемого "нелинейного коэффициента безопасности" напряжений. Кроме того, разность относительных уменьшений больше для более широких спектральных плотностей нагрузки.